The generalized Euler-Poinsot rigid body equations: explicit elliptic solutions*

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Abstract

The classical Euler–Poinsot case of the rigid body dynamics admits a class of simple but non-trivial integrable generalizations, which modify the Poisson equations describing the motion of the body in space. These generalizations possess first integrals which are polynomial in the angular momenta.

We consider the modified Poisson equations as a system of linear equations with elliptic coefficients and show that all the solutions of it are single-valued. By using the vector generalization of the Picard theorem, we derive the solutions explicitly in terms of sigma functions of the corresponding elliptic curve. The solutions are accompanied with a numerical example. We also compare the generalized Poisson equations with the classical 3rd order Halphen equation.

1 Introduction

As in [1], we consider the following system

$$J\dot{\omega} = J\omega \times \omega, \quad \dot{\gamma} = \gamma \times B\omega, \qquad \omega, \gamma \in \mathbb{R}^3, \tag{1.1}$$

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which is a certain limit of the Kirchhoff equations describing the motion of a rigid body in an ideal fluid. Here ω is the angular velocity of the body, γ is the linear momentum; 3×3 matrices J, and B are tensors of adjoint masses. The first equation in (1.1) are just the Euler equations describing the free motion of the body with the inertia tensor J.

In the sequel, J and B are assumed to be arbitrary diagonal matrices. In the special case $B = \mathrm{Id}_3$, the system becomes the classical integrable Euler-Poinsot case of the rigid body motion. In this case vector γ is a vertical vector fixed in space. Then, in the general case, ω is solved in terms of elliptic functions, and the 3 independent solutions for the vector γ are elliptic functions and elliptic functions of the second kind, see e.g., [4, 11].

Setting $M = J\omega$, the system (1.1) can be rewritten in the form

$$M = M \times aM, \qquad \dot{\gamma} = \gamma \times bM, \tag{1.2}$$

where

$$a = \mathrm{diag}(a_1, a_2, a_3) := J^{-1}, \qquad b = \mathrm{diag}(b_1, b_2, b_3) := BJ^{-1}. \tag{1.3}$$

It has three independent polynomial first integrals

$$H_1 := \langle M, aM\rangle, \qquad H_2 := \langle M, M\rangle, \qquad H_3 := \langle \gamma, \gamma\rangle, \tag{1.4}$$

Here and below $\langle x, y\rangle$ denotes the scalar product of vectors x, $y \in \mathbb{R}^3$. As the system (1.2) is divergence free, according to the Euler–Jacobi theorem, for its integrability one additional first integral is required.

In [1], the authors applied the Kovalevskaya–Painlevé method to search for integrable cases of the considered system. It was shown that if all the solutions of the system (1.2) are meromorphic, or single-valued, then

$$k^2 a_{32}a_{13}a_{21} + b_1^2 a_{32} + b_2^2 a_{13} + b_3^2 a_{21} = 0, \quad a_{ij} = a_i - a_j, \tag{1.5}$$

where k is an *odd* integer. Geometrically, the above condition describes a quadric in $\mathbb{R}^3$ with coordinates (b_1, b_2, b_3). In particular, one has $b = ka$. In this case the equations (1.2) become what can be called the modified Euler-Poinsot system

$$\dot{M} = M \times aM, \quad \dot{\gamma} = k\,\gamma \times aM. \tag{1.6}$$

As it was shown in [1], if the condition (1.5) is satisfied, then for *odd positive* k the system (1.2) possess an additional first integral H_4, which is algebraically independent with (1.4). It is linear in γ, and of degree k in M, and can be written in the following form

$$H_4 = \langle P(M), \gamma\rangle, \tag{1.7}$$

where the vector $P(M)$ is given by

$$P(M) = \mathrm{diag}(M_1, M_2, M_3)\Phi_k(M)\,T, \tag{1.8}$$

$$\Phi_k(M) := \left(A_1^{-1}K\right) \cdot \left(A_3^{-1}K\right) \cdots \left(A_{k-2}^{-1}K\right). \tag{1.9}$$

The matrices K, A_n are defined as follows

$$K = \operatorname{diag}\left(M_1^2, M_2^2, M_3^2\right), \qquad A_n = \begin{pmatrix} -n\,a_{32} & b_3 & -b_2 \\ -b_3 & -n\,a_{13} & b_1 \\ b_2 & -b_1 & -n\,a_{21} \end{pmatrix}, \quad n \in \mathbb{N}.$$

The constant vector $T \in \mathbb{R}^3$ in formula (1.8) spans the kernel of the matrix A_k.

Notice that

$$\det A_n = -n(n^2 a_{32}a_{13}a_{21} + b_1^2 a_{32} + b_2^2 a_{13} + b_3^2 a_{21}).$$

As will be shown in Section 8, the case of negative odd k can be reduced to the above one.

In the simplest non-trivial case $b = ka$ with $k = 3$, we have

$$P(M) = P^{(3)} := \left(P_1^{(3)}, P_2^{(3)}, P_3^{(3)}\right)^T \tag{1.10}$$

with

$$\begin{aligned} P_1^{(3)} &= M_1\,[(a_1a_2 + 8a_1^2 - a_3a_2 + a_3a_1)M_1^2 \\ &\qquad + (3a_3a_2 - 3a_3a_1 + 9a_1a_2)M_2^2 + (9a_3a_1 - 3a_1a_2 + 3a_3a_2)M_3^2], \\ P_2^{(3)} &= M_2\,[(-3a_3a_2 + 3a_3a_1 + 9a_1a_2)M_1^2 \\ &\qquad + (a_3a_2 - a_3a_1 + 8a_2^2 + a_1a_2)M_2^2 + (3a_3a_1 - 3a_1a_2 + 9a_3a_2)M_3^2], \\ P_3^{(3)} &= M_3\,[(9a_3a_1 + 3a_1a_2 - 3a_3a_2)M_1^2 \\ &\qquad + (-3a_3a_1 + 3a_1a_2 + 9a_3a_2)M_2^2 + (a_3a_1 + 8a_3^2 - a_1a_2 + a_3a_2)M_3^2]. \end{aligned}$$

In [1] it was also shown that the vector $P(t) := P(M(t))$, where $M(t)$ is a solution of the Euler equation in (1.6), itself is a meromorphic solution of the Poisson equation in (1.6). Since the solution $M(t)$ in terms of elliptic or hyperbolic functions is well-known, the solution $P(t)$ can be found by using (1.8).

In the sequel we will regard the generalized Poisson equations in (1.6) as a separate system of linear equations

$$\dot{\gamma} = \mathcal{A}(t)\gamma, \qquad \mathcal{A} = k \begin{pmatrix} 0 & a_3M_3 & -a_2M_2 \\ -a_3M_3 & 0 & a_1M_1 \\ a_2M_2 & -a_1M_1 & 0 \end{pmatrix} \in \mathrm{so}(3). \tag{1.11}$$

with the coefficients given by the elliptic functions $M(t)$. (We will not consider the special cases when $M(t)$ are hyperbolic functions describing asymptotic motions of the Euler top.)

In the present paper we show that under the condition $b = ka$ (k is odd), all the solutions of (1.11) are meromorphic. Our main goal is to give an explicit form of their three independent complex solutions in terms of elliptic functions and elliptic functions of the second kind (sigma-functions and exponents), as presented in Theorems 4 and 6 below.

Additionally, in Theorem 7, we give expressions for the components of the associated real orthogonal rotation matrix $\mathcal{R}(t)$ whose columns satisfy the Poisson equations.

These equations give rise to a 3rd order ODE for one of the components of the vector γ. In the final part, we compare this ODE with the best known integrable ODE with elliptic coefficients, namely the Halphen equation, and show that, in general, they cannot be transformed into eauch other.

2 General properties of the solutions

As was already mentioned, the Kovalevskaya–Painlevé analysis made in [1] shows that for all the solutions of the system (1.2) or the Poisson equation (1.11) to be single-valued, the condition (1.5) must hold and k must be an odd integer. We show that these conditions are also sufficient[1].

Lemma 1. *For an arbitrary solution $M_1(t), M_2(t), M_3(t)$ of the Euler equations, all solutions of generalized Poisson equations* (1.11) *are single-valued if and only if k is an odd integer and condition* (1.5) *is fulfilled.*

Proof. An elliptic or a hyperbolic solution $M(t)$ of the Euler equation has four simple poles in the fundamental region. Hence, all singular points of equation (1.11) on $\mathbb{C}$ are regular. Since the equation is linear, branching of its solutions can happen only at the singular points.

If k is an odd integer and condition (1.5) is satisfied, then all exponents at each singular point are integers. However a branching can still occur if the local series solution in a neighborhood of a singular point has logarithmic terms. We will show that this never happens due to the presence of the first integral (1.7). Namely, the integral implies that the equation (1.11) has time dependent first integral $I_4(t,\gamma) := \langle P(t), \gamma\rangle$, which is polynomial of degree k in γ, and $P(t)$ is the corresponding elliptic solution of (1.11). Assume that $P(t)$ is normalized: $\langle P(t), P(t)\rangle = 1$.

Now take $t_0 \in \mathbb{C}$ which does not coincide with a pole of $M(t)$, and a loop $s \mapsto \tau(s) \in \mathbb{C}$, $s \in [0,1]$, $\tau(0) = \tau(1) = t_0$, which encircles once counterclockwise a pole t^* of $M(t)$. Let $\Gamma(t)$ be a fundamental matrix of (1.11) with the first column proportional to $P(t)$ and let $\Gamma(t_0) \in \mathrm{SO}(3,\mathbb{C})$. Then $\Gamma(t) \in \mathrm{SO}(3,\mathbb{C})$ for all t where it is defined.

A continuation along the loop τ gives a monodromy matrix $\mathcal{M}_\tau \in \mathrm{SO}(3,\mathbb{C})$:

$$\Gamma(\tau(s+1)) = \Gamma(\tau(s))\,\mathcal{M}_\tau .$$

For any solution $\gamma(t) = \Gamma(t)\vec{v}$, $\vec{v} = \text{const} \in \mathbb{C}^3$, the integral $I_4(t,\gamma)$ implies

$$\langle P(t_0), \gamma(t_0)\rangle = \langle P(t_0), \Gamma(t_0)\vec{v}\rangle = \langle P(t_0), \Gamma(t_0)\mathcal{M}_\tau\vec{v}\rangle .$$

Since $\vec{v}$ is arbitrary, this yields $P^T(t_0)\Gamma(t_0) = P^T(t_0)\Gamma(t_0)\mathcal{M}_\tau$ and, due to the ortogonality of $\Gamma(t)$ and the normalization of $P(t)$,

$$(1,0,0) = (1,0,0)\mathcal{M}_\tau \, .$$

[1] In fact, this was already stated in [1], however, without a proof.

Then, since, $\mathcal{M}_\tau$ is also orthogonal, it must have the block structure

$$\mathcal{M}_\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & \vartheta \\ 0 & -\vartheta & \theta \end{pmatrix}, \qquad \theta, \vartheta \in \mathbb{C}, \quad \theta^2 + \vartheta^2 = 1. \tag{2.1}$$

We now recall that the formal series solution in a neighborhood of the singular point t^* has logarithmic terms if and only if the monodromy matrix $\mathcal{M}_\tau$ is not diagonalizable (see, e.g., [2]). However (2.1) is diagonalizable for any θ, ϑ satisfying the above condition. $\square$

One of the main tools of our subsequent analysis will be a vector extension of the known Picard theorem formulated, in particular, in [6, 7]. For our purposes we adopt it in the following form.

Theorem 2. *Let T_1 and T_2 be the common, real and imaginary periods of the elliptic solutions $M_1(t), M_2(t), M_3(t)$ of the Euler equations. If all the solutions of (1.11) are meromorphic, then, apart from the elliptic vector solution $\gamma(t) = P(M(t))$ of (1.11), there exist two elliptic solutions of the second kind $\gamma(t) = G^{(1)}(t)$, and $\gamma(t) = G^{(2)}(t)$, which satisfy*

$$G^{(1)}(t + T_j) = S_j G^{(1)}(t), \quad G^{(2)}(t + T_j) = S_j^{-1} G^{(2)}(t), \qquad j = 1, 2, \tag{2.2}$$

where $S_1, S_2 \in \mathbb{C}$, and, moreover, $|S_1| = 1$.

Proof. The existence of at least one vector solution of the second kind, $G(t)$, follows from the vector extension of the Picard theorem mentioned above. Let s_1, s_2 be its monodromy factors with respect to the periods T_1, T_2.

Let $\widehat{G}(t)$ be another solution of (1.11), and $\Gamma(t) = \Big(G(t), \widehat{G}(t), P(t)\Big)$ be a fundamental matrix. The monodromy matrices $\mathcal{M}_1$, and $\mathcal{M}_2$, corresponding to periods T_1, T_2, respectively, are given by

$$\Gamma(t + T_j) = \Big(s_j G,\ \chi_j G + \hat{\chi}_j \hat{G} + \rho_j P,\ P\Big) = \Gamma(t)\mathcal{M}_j,$$

where

$$\mathcal{M}_j = \begin{pmatrix} s_j & \chi_j & 0 \\ 0 & \hat{\chi}_j & 0 \\ 0 & \rho_j & 1 \end{pmatrix}, \quad j = 1, 2,$$

where $\chi_j, \hat{\chi}_j, \rho_j$ are certain constants. Observe that, regardless to the values of the constants, both monodromy matrices $\mathcal{M}_1$, and $\mathcal{M}_2$ are diagonalizable.

Next, since, by the assumption, all the solutions of (1.11) are meromorphic, the monodromy group must be trivial. Therefore, $\mathcal{M}_1$, and $\mathcal{M}_2$ commute, and are diagonalizable in the same basis. As a result, there exist two independent solutions of the second kind $G^{(1)}(t), G^{(2)}(t)$ forming the fundamental matrix $\big(G^{(1)}(t), G^{(2)}(t), P(t)\big)$. Following the general Floquiet theory, the corresponding monodromy matrices $\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2$ must satisfy

$$\det \bar{\mathcal{M}}_j = \exp\left(\int_0^{T_j} \operatorname{Tr} \mathcal{A}(t) dt\right) = 1, \qquad j = 1, 2.$$

(Here we used the property $\mathcal{A}(t) \in so(3,\mathbb{C})$.) Hence, since the monodromy of the elliptic solution $P(t)$ is trivial, the monodromy factors of $G^{(1)}(t)$, and $G^{(2)}(t)$ are reciprocal, and this implies (2.2).

Further, let for certain constants ν_1, $\nu_2 \in \mathbb{C}$

$$\gamma(t) = \nu_1 G^{(1)}(t) + \nu_2 G^{(2)}(t), \quad t \in \mathbb{R}$$

be a real vector solution of the Poisson equation. This means that, for $i = 1,2,3$, the numbers $\nu_1 G_i^{(1)}(t)$ and $\nu_2 G_i^{(2)}(t)$ are complex conjugated. Then, for the real period T_1, the vector $\gamma(t+T_1)$ is also a real solution. On the other hand, from the above and from the monodromy (2.2), we deduce that

$$\gamma_i(t+T_1) = S_1\, \nu_1 G_i^{(1)}(t+T_1) + S_1^{-1}\, \nu_2 G_i^{(2)}(t+T_1),$$

which is real if and only if $|S_1| = 1$. $\square$

3 Algebraic parametrization and elliptic sigma-function solution for M and $P(M)$.

We first recall how generic solutions of the Euler equation in (1.2) can be expressed in terms of the Weierstrass sigma functions. We need this fact to derive the general solution of the Poisson equation.

Let us fix a common level of first integrals (1.4)

$$\langle M, aM\rangle = l, \qquad \langle M, M\rangle = m^2, \qquad \langle \gamma, \gamma\rangle = 1 \tag{3.1}$$

For a generic values χ, m, solutions $M_i(t)$ of the Euler equations are elliptic functions related to the elliptic curve E, given by

$$E = \left\{\mu^2 = U_4(\lambda)\right\}, \qquad U_4(\lambda) := -(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c), \tag{3.2}$$

where $c := l/m^2$. Here and below we assume that $c \neq a_1, a_2, a_3$. This curve, compactified and regularized, has two infinite points $\infty_\pm$.

A "rational" parametrization of the momenta M_i in terms of the coordinates λ, see, e.g., [5], have the following form

$$M_\alpha = m\sqrt{\frac{(a_\beta - c)(a_\gamma - c)}{(a_\alpha - a_\beta)(a_\alpha - a_\gamma)}}\sqrt{\frac{\lambda - a_\alpha}{\lambda - c}}. \tag{3.3}$$

Then, from the Euler equations, we easily deduce that the evolution of λ is given by the equation

$$\dot\lambda = 2m\sqrt{-(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c)}. \tag{3.4}$$

That is, for any $\lambda \in \mathbb{C}$, the right hand sides of (3.3) satisfy the equations (3.1).

For a real motion, i.e., for real values of l, m, and t, if $a_1 < a_2 < a_3$, then one has $c \in (a_1, a_3)$, $c \neq a_2$. Moreover,

$$\lambda \in \begin{cases} [a_2, a_3], & \text{if} \quad a_1 < c < a_2, \\ [a_1, a_2], & \text{if} \quad a_2 < c < a_3. \end{cases}$$

The birational map $(\lambda,\mu)\to(z,w)$, given by

$$z=\frac{1}{3}\frac{(\tau_2-2c\tau_1+3c^2)\lambda+2c\tau_2-c^2\tau_1-3\tau_3}{\lambda-c},\quad w=\frac{\mu}{(\lambda-c)^2}, \tag{3.5}$$

transforms the elliptic curve E to its canonical Weierstrass form

$$\mathcal{E}=\left\{w^2=U_3(z)\right\},\qquad U_3(z):=4(z-e_1)(z-e_2)(z-e_3)=4z-g_2z-g_3 \tag{3.6}$$

where

$$\begin{gathered}3e_\alpha=\tau_2+c\tau_1-3(a_\beta a_\gamma+ca_\alpha),\qquad e_1+e_2+e_3=0,\\ \tau_1=a_1+a_2+a_3,\qquad \tau_2=a_1a_2+a_2a_3+a_3a_1,\qquad \tau_3=a_1a_2a_3.\end{gathered} \tag{3.7}$$

The above map sends $\lambda=c$ to $z=\infty$, and a_i to e_i, respectively. Then there is the following relation between the holomorphic differentials on E and $\mathcal{E}$:

$$\mathrm{i}\frac{\mathrm{d}\lambda}{2\sqrt{U_4(\lambda)}}=\frac{\mathrm{d}z}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}}.$$

Let us introduce the Abel map

$$u=\mathrm{i}\int_c^p\frac{\mathrm{d}\lambda}{2\sqrt{U_4(\lambda)}},\quad\text{where}\quad p=(\lambda,\mu)\in E. \tag{3.8}$$

The integrals

$$\Omega_\alpha:=\mathrm{i}\int_c^{a_\alpha}\frac{\mathrm{d}\lambda}{2\sqrt{U_4(\lambda)}},\qquad \alpha=1,2,3$$

are the half-periods of the curve E. We choose the sign of the root $U_4(\lambda)$ to ensure $\Omega_1+\Omega_2+\Omega_3=0$.

According to (3.8), in the case $a_1<c<a_2<a_3$ the half-period Ω_1 is imaginary and Ω_2 is real, whereas for $a_1<a_2<c<a_3$ the half-period Ω_3 is imaginary and Ω_2 is real. In both cases, comparing (3.8) with (3.4), we get

$$u=\mathrm{i}m(t-t_0)+\Omega_2. \tag{3.9}$$

Using the Weierstrass sigma function $\sigma(u)=\sigma(u|\,2\Omega_1,2\Omega_3)$, one can write

$$\lambda-c=\text{const}\;\cdot\frac{\sigma^2(u)}{\sigma(u-h)\,\sigma(u+h)},\quad\text{where}\quad h=\int_c^\infty\frac{d\lambda}{\sqrt{U_4(\lambda)}}, \tag{3.10}$$

$$\frac{\lambda-\rho}{\lambda-c}=\text{const}\;\cdot\frac{\sigma(u-\beta)\sigma(u+\beta)}{\sigma^2(u)},\quad \beta=\int_c^\rho\frac{\mathrm{d}\lambda}{\sqrt{U_4(\lambda)}}. \tag{3.11}$$

Moreover, we also have

$$\sqrt{\frac{\lambda-a_\alpha}{\lambda-c}}=C_\alpha\frac{\sigma_\alpha(u)}{\sigma(u)},\quad \alpha=1,2,3, \tag{3.12}$$

where C_α are certain constants and $\sigma_\alpha(u)$ are the sigma-functions obtained from $\sigma(u)$ by shift of u, and by multiplication by an exponent:

$$\sigma_\alpha(u) := \mathrm{e}^{\eta_\alpha u}\frac{\sigma(\Omega_\alpha - u)}{\sigma(\Omega_\alpha)}, \qquad \eta_\alpha = \zeta(\Omega_\alpha), \quad \zeta(u) = \frac{\sigma'(u)}{\sigma(u)}, \tag{3.13}$$

where $\alpha = 1,2,3$. Note that we have

$$\sigma(u) = u - \frac{g_2}{240}u^5 - \frac{g_3}{840}u^7 + \cdots, \quad \text{and} \quad \sigma_1(0) = \sigma_2(0) = \sigma_3(0) = 1. \tag{3.14}$$

see, e.g., [8] or [9]. From (3.3), (3.12), it follows that the solutions of the Euler equations have the form

$$M_\alpha = h_\alpha \frac{\sigma_\alpha(u)}{\sigma(u)}, \quad u = \mathrm{i}m\,t + \Omega_2, \quad \alpha = 1,2,3. \tag{3.15}$$

with certain constants h_α which we determine below. In view of (3.13), the sigma-quotients have the quasiperiodic property

$$\frac{\sigma_\alpha(u+2\Omega_j)}{\sigma(u+2\Omega_j)} = (-1)^{1-\delta_{\alpha,j}}\frac{\sigma_\alpha(u)}{\sigma(u)}, \tag{3.16}$$

where $\delta_{\alpha,j}$ is the Kronecker symbol. Hence, the coefficients $M_\alpha(u)$ of the Poisson equation (1.11) have common periods $4\Omega_1, 4\Omega_2$.

Next, using the parametrization (3.3), and the expressions (1.8), for each odd k we get the following parametrization for the elliptic solution $P(M) = (P_1, P_2, P_3)^T$:

$$P_\alpha = \sqrt{\frac{(a_\beta - c)(a_\gamma - c)}{(a_\alpha - a_\beta)(a_\alpha - a_\gamma)}}\sqrt{\frac{\lambda - a_\alpha}{\lambda - c}}\frac{F_{s,\alpha}(\lambda)}{(\lambda - c)^s}, \tag{3.17}$$

where $F_{s,\alpha}(\lambda) = \rho_{0,\alpha}\prod_{r=1}^{s}(\lambda - \rho_{r,\alpha})$ is a polynomial of degree $s = (k-1)/2$, which is obtained by substituting (3.3) into the vector $\Phi_k T$ in (1.8) and taking the numerator. The sum $\Delta_k = P_1^2(\lambda) + P_2^2(\lambda) + P_2^2(\lambda)$ is a constant depending on a_α, and c only.

In particular, for $k = 3$, and $b_\alpha = 3a_\alpha$, by using (1.10), we have

$$\begin{aligned}
F_{11}(\lambda) &= [3\tau_2 + 4c(c - \tau_1 - 2a_1) - 2a_2a_3]\lambda + c(\tau_2 + 2a_2a_3) - 4\tau_3 + 8c^2a_1, \\
F_{12}(\lambda) &= [3\tau_2 + 4c(c - \tau_1 - 2a_2) - 2a_3a_1]\lambda + c(\tau_2 + 2a_3a_1) - 4\tau_3 + 8c^2a_2, \\
F_{13}(\lambda) &= [3\tau_2 + 4c(c - \tau_1 - 2a_3) - 2a_1a_2]\lambda + c(\tau_2 + 2a_1a_2) - 4\tau_3 + 8c^2a_3
\end{aligned} \tag{3.18}$$

and $\Delta_3 = \tau_2^2 - 4\tau_1\tau_3 + 36c\tau_3 - 48c^2\tau_2 + 64c^3\tau_1$.

Now, applying expressions (3.11), (3.12) to (3.17), we get

$$P_\alpha = c_\alpha \frac{\sigma_\alpha(u)}{\sigma(u)}\prod_{r=1}^{s}\frac{\sigma(u + v_{r,\alpha})\sigma(u - v_{r,\alpha})}{\sigma^2(u)}, \tag{3.19}$$

where

$$v_{r,\alpha} = \pm\mathrm{i}\int_c^{\rho_{r,\alpha}}\frac{\mathrm{d}\lambda}{2\sqrt{U_4(\lambda)}}, \qquad u = \mathrm{i}mt + \Omega_2,$$

for $r = 1, \ldots, s$. Thus, the components $P_\alpha(u)$ have a pole of order k at $u = 0$ and, like $M_\alpha(u)$, they are doubly periodic with common periods $4\Omega_1, 4\Omega_2$. We finally have

Proposition 3. *The momentum vector M and the elliptic vector solution P of the Poisson equations can be written as*

$$M_\alpha = m\epsilon_\alpha \frac{\sigma_\alpha(u)}{\sigma(u)}, \quad P_\alpha = \epsilon_\alpha \frac{\sigma_\alpha(u)}{\sigma(u)} \prod_{r=1}^{s} \frac{\sigma(u+v_{r,\alpha})\,\sigma(u-v_{r,\alpha})}{\sigma^2(v_{r,\alpha})\,\sigma^2(u)}, \tag{3.20}$$

$$\epsilon_\alpha = \frac{1}{\sqrt{(a_\alpha - a_\beta)(a_\alpha - a_\gamma)}}, \quad (\alpha,\beta,\gamma) = (1,2,3), \tag{3.21}$$

where signs of ϵ_α are chosen according to the condition

$$\frac{1}{\epsilon_1\epsilon_2\epsilon_3} = -(a_1-a_2)(a_2-a_3)(a_3-a_1),$$

and u depends on time t via (3.9).

Then also

$$P_\alpha^2(u) = \epsilon_\alpha \left(\wp(u) - \wp(\Omega_\alpha)\right) \prod_{l=1}^{s} \left(\wp(u) - \wp(v_{l,\alpha})\right)^2, \tag{3.22}$$

where $\wp(u) = \wp(u|g_2,g_3)$ is the Weierstrass P-function.

Here, for any $u \in \mathbb{C}$

$$M_1^2(u) + M_2^2(u) + M_3^2(u) = m^2, \quad P_1^2(u) + P_2^2(u) + P_3^2(u) = \Pi, \tag{3.23}$$

$$\begin{aligned}\Pi = {} & \epsilon_\beta^2(\wp(\Omega_\alpha) - \wp(\Omega_\beta)) \prod_{l=1}^{s} \left(\wp(\Omega_\alpha) - \wp(v_{l,\beta})\right)^2 \\ & + \epsilon_\gamma^2(\wp(\Omega_\alpha) - \wp(\Omega_\gamma)) \prod_{l=1}^{s} \left(\wp(\Omega_\alpha) - \wp(v_{l,\gamma})\right)^2,\end{aligned} \tag{3.24}$$

for any permutation $(\alpha,\beta,\gamma) = (1,2,3)$.

Remark. According to the rule (3.16), the shift $u \to u + 2\Omega_\alpha$ in the solutions (3.20) is equivalent to flip of signs of some of the constants ϵ_i in such a way that the above condition is satisfied.

Proof of Proposition 3. To calculate the constants h_α, c_α in the elliptic solutions (3.15), (3.19), we compare the leading terms of their Laurent expansions near the poles and the expansions of the sigma functions. Namely, let $t_0 \in \mathbb{C}$ be a pole of the functions $M(t), P(t)$, and $\delta t = t - t_0$. Substituting

$$M = \frac{1}{\delta t}\left(M^{(0)} + M^{(1)}\delta t + \cdots\right), \quad P = \frac{1}{(\delta t)^k}\left(P^{(0)} + P^{(1)}\delta t + \cdots\right)$$

into the equations (1.6) for M and γ, for any $k \in \mathbb{N}$, one gets

$$M^{(0)} \in \left\{ \mathrm{i}(\epsilon_1,\epsilon_2,\epsilon_3)^T,\ \mathrm{i}(-\epsilon_1,-\epsilon_2,\epsilon_3)^T,\ \mathrm{i}(-\epsilon_1,\epsilon_2,-\epsilon_3)^T,\ \mathrm{i}(\epsilon_1,-\epsilon_2,-\epsilon_3)^T \right\}, \tag{3.25}$$

with ϵ_α given by (3.21), and $P^{(0)}$ is proportional to $M^{(0)}$.

On the other hand, in view of (3.14), near $u=0$ we have the expansions

$$\frac{\sigma_\alpha(u)}{\sigma(u)} = \frac{1}{u} + O(1),$$
$$\prod_{r=1}^{s} \frac{\sigma(u+v_{r,\alpha})\sigma(u-v_{r,\alpha})}{\sigma^2(u)} = -\frac{\sigma^2(v_{1,\alpha})\cdots\sigma^2(v_{s,\alpha})}{u^{k-1}} + O(1).$$

with $s=(k-1)/2$. Since in the above expansions $u = \mathrm{i}m\cdot\delta t$, comparing them, we obtain[2]

$$h_\alpha = m\epsilon_\alpha, \quad c_\alpha = \frac{\epsilon_\alpha}{\sigma^2(v_1)\cdots\sigma^2(v_s)}.$$

Substituting this into (3.15), (3.19), we get (3.20).

The latter, in view of the known relations (see, e.g., [8, 9])

$$\frac{\sigma_\alpha^2(u)}{\sigma^2(u)} = \wp(u) - \wp(\Omega_\alpha), \quad \frac{\sigma(u+\beta)\sigma(u-\beta)}{\sigma^2(\beta)\,\sigma^2(u)} = \wp(u) - \wp(\beta),$$

implies (3.22).

Finally, since $P(u)$ is a solution of the Poisson equations, it satisfies the integral (3.23). Setting there $u=\Omega_\alpha$ and using (3.22) one obtains (3.24). □

Remark. As follows from the formal Laurent solution for $M(t)$ with the coefficients (3.25), near a pole $t=t_0$ *any* vector solution $\gamma(t) = (\gamma_1,\gamma_2,\gamma_3)^T$ of the Poisson equation (1.11) has the expansion

$$\gamma_\alpha(t) = \frac{\text{const}}{(\delta t)^k}\left(\epsilon_\alpha + O(\delta t)\right), \quad \alpha = 1,2,3. \tag{3.26}$$

4 Algebraic structure of elliptic solutions of 2nd kind

Using the algebraic parameterizations (3.3) and (3.17), we obtain

Theorem 4. *1) If k is a positive odd integer and $k\geq 3$, then, apart from the solution $P(\lambda)$ in* (3.17)*, the Poisson equations* (1.11) *has two independent solutions*

$$\gamma^{(j)} = G^{(j)} = \left(G_1^{(j)}, G_2^{(j)}, G_3^{(j)}\right)^T, \qquad j=1,2$$

which can be represented as the following algebraic functions of the parameter λ in (3.3)*,* (3.4)

$$\begin{aligned} G_\alpha^{(1)} &= c_{1,\alpha}\frac{\sqrt{Q_{k,\alpha}(\lambda)}}{\sqrt{(\lambda-c)^k}}\exp\left(\frac{1}{2}\int W_\alpha\right), \\ G_\alpha^{(2)} &= c_{2,\alpha}\frac{\sqrt{Q_{k,\alpha}(\lambda)}}{\sqrt{(\lambda-c)^k}}\exp\left(-\frac{1}{2}\int W_\alpha\right).i \end{aligned} \tag{4.1}$$

[2] If fact, one can write h_α, c_α only in terms of sigma-constants and $\sigma(v_j)$, as it was written for the Euler top (the case $k=1$) (see [4, 11]), but this process is tedious and requires more calculations.

Here $c_{1,\alpha}, c_{2,\alpha}$ are certain constants to be specified below, and

$$W_\alpha = \frac{q_{s+1,\alpha}(\lambda)\cdot(\lambda-c)^s}{Q_{k,\alpha}(\lambda)}\,\frac{\mathrm{d}\lambda}{\sqrt{U_4(\lambda)}}, \tag{4.2}$$

$$\begin{aligned} Q_{k,\alpha}(\lambda) &= r_{0,\alpha}\prod_{i=1}^{k}(\lambda - r_{i,\alpha}) = \mathrm{const}\cdot(P_\beta^2(\lambda)+P_\gamma^2(\lambda))\cdot(\lambda-c)^k \\ &= (a_\beta - c)(a_\gamma - a_\beta)(\lambda - a_\beta)F_{s,\beta}^2(\lambda) + (a_\gamma - c)(a_\gamma - a_\beta)(\lambda - a_\gamma)F_{s,\gamma}^2(\lambda), \quad (4.3)\\ &\qquad (\alpha,\beta,\gamma) = (1,2,3), \end{aligned}$$

where the polynomials $F_{s,\alpha}(\lambda)$ of degree $s=(k-1)/2$ are specified in (3.17).

The above formula implies that the zeros of the polynomials $Q_{k,\alpha}(\lambda)$ coincide with the zeros of $P_\beta^2(\lambda)+P_\gamma^2(\lambda)$.

2) The differential W_α is a meromorphic differential of the third kind on $\mathcal{E}$ having pairs of only simple poles at the points $\mathcal{P}_{i,\alpha}^{\pm} = (r_{i,\alpha}, \pm\sqrt{U_4(r_{i,\alpha})}) \in \mathcal{E}$, $i=1,\dots,k$ with residues ± 1 respectively:

$$\mathrm{Res}_{\mathcal{P}_{i,\alpha}^{\pm}} W_\alpha = \pm 1, \qquad i=1,\dots,k. \tag{4.4}$$

Finally, $q_{s+1,\alpha}(\lambda)$ in (4.2) *are polynomials of degree $s+1=(k+1)/2$ completely defined by the conditions* (4.4)

The algebraic solutions in the classical case $k=1$ will be described separately in Section 6.

Remark. The polynomials $F_{s,1}, F_{s,2}, F_{s,3}$ and $Q_{k,1}, Q_{k,2}, Q_{k,3}$ are obtained by the corresponding permutation of a_1, a_2, a_3. Note that their Abel images (u-coordinates) of their roots $\rho_{r,\alpha}, r_{i,\alpha}$ are not obtained from each other by the translations by the half-periods Ω_j of the elliptic curve E.

Proof of Theorem 4. 1) According to the kinematic interpretation, the Poisson equations in (1.6) describes the evolution of a fixed in the space vector γ in a frame rotating with the angular velocity $\tilde\omega = BM$ (also taken in the body frame). Now choose a fixed in space ortonormal frame $\{O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Let θ, ψ, ϕ be the Euler nutation, precession, and rotation angles associated to this frame so that the corresponding rotation matrix is

$$\mathcal{R} = \begin{pmatrix} \cos\phi\cos\psi - \cos\theta\sin\psi\sin\phi & \cos\phi\sin\psi + \cos\theta\cos\psi\sin\phi & \sin\phi\sin\theta \\ -\sin\phi\cos\psi - \cos\theta\sin\psi\cos\phi & -\sin\phi\sin\psi + \cos\theta\cos\psi\cos\phi & \cos\phi\sin\theta \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{pmatrix}.$$

Let $\vec{P}(t) = (P_1, P_2, P_3)^T$ be a solution of (1.6) describing the motion of the vector $|P|\mathbf{e}_3$. Then, in view of the structure of $\mathcal{R}$, and from the Euler kinematic equations, one has

$$P_1 = |P|\sin\phi\sin\theta, \quad P_2 = |P|\cos\phi\sin\theta, \quad P_3 = |P|\cos\theta,$$

and

$$\dot\psi = |P|\frac{\tilde\omega_1 P_1 + \tilde\omega_2 P_2}{P_1^2 + P_2^2}, \tag{4.5}$$

see e.g., [11]. Hence the thirds components of the other two independent solutions $G^{(1)}$, and $G^{(2)}$ of (1.11) can be written in the complex form

$$G_3^{(j)} = \tilde{c}_j \sin\theta \exp(\pm \mathrm{i}\psi), \quad \text{or} \quad G_3^{(j)} = c_j \sqrt{P_1^2 + P_2^2} \exp\left(\pm \mathrm{i} \int \dot{\psi}\, \mathrm{d}t \right), \tag{4.6}$$

$\tilde{c}_j$, and c_j are certain constants, and $j = 1, 2$. Using the parametrization (3.17) for $P_\alpha(\lambda)$, as well as relations (3.3) and (3.4), we get

$$\mathrm{i}\dot{\psi}\, dt = \mathrm{i}\sqrt{\Delta_k} k \frac{a_1 M_1(\lambda) P_1(\lambda) + a_2 M_2(\lambda) P_2(\lambda)}{P_1^2(\lambda) + P_2^2(\lambda)} \frac{d\lambda}{2m\sqrt{U_4(\lambda)}} := \frac{1}{2} W_3. \tag{4.7}$$

After simplifications this takes the form

$$\mathrm{i}\dot{\psi}\, dt = \frac{1}{2} \frac{q_{s+1,3}(\lambda) \cdot (\lambda - c)^s}{Q_{k,3}(\lambda)} \frac{d\lambda}{\sqrt{U_4(\lambda)}},$$

with

$$\begin{aligned} q_{s+1,3}(\lambda) &= \frac{\mathrm{i}\sqrt{\Delta_k}\, k}{m} [(a_2 - c)(a_3 - a_2) b_1 \cdot (\lambda - a_1) F_{s,1}(\lambda) \\ &\qquad\qquad + (a_1 - c)(a_1 - a_3) b_2 \cdot (\lambda - a_2) F_{s,2}(\lambda)], \\ Q_{k,3}(\lambda) &= (a_2 - c)(a_3 - a_2) \cdot (\lambda - a_1) F_{s,1}^2(\lambda) + (a_1 - c)(a_1 - a_3) \cdot (\lambda - a_2) F_{s,2}^2(\lambda), \\ \sqrt{P_1^2 + P_2^2} &= \text{const}\, \frac{\sqrt{Q_{k,3}(\lambda)}}{\sqrt{(\lambda - c)^k}}. \end{aligned}$$

The above implies the formulas (4.1)–(4.3) for $\alpha = 3$. Repeating the same geometric argumentation for $\alpha = 1, 2$, we get the whole set of formulas of Theorem 4.

2) The differential W_α in (4.2) has simple poles at $\lambda \in \{r_{1,\alpha}, \ldots, r_{k,\alpha}\}$, and each of them corresponds to two points $\mathcal{P}_{i,\alpha}^{\pm}$ on E. In view of the degrees of polynomials $Q_{k,\alpha}(\lambda)$, and $q_{s+1,\alpha}$, this differential does not have poles at the infinite points $\infty_\pm$ on E. Next, $\mathrm{d}\lambda/\sqrt{U_4(\lambda)}$ does not have neither poles nor zeros on E. Hence W_α is a differential of the third kind.

Next, the residuum conditions (4.4) are necessary for the solutions (4.1) to be meromorphic in t, or u and, locally, in λ. Namely, let $\tau = \lambda - r_{i,\alpha}$ be a local coordinate on E near the root $r_{i,\alpha}$ and the meromorphic differentials have the expansion

$$W_\alpha = \left(\frac{\varkappa}{\tau} + O(1) \right) \mathrm{d}\tau.$$

Assume $\varkappa > 0$. Then, as follows from (4.6) for $\alpha = 3$, the leading term of the expansion of the solution Γ_3 has the form

$$\text{const} \cdot \sqrt{\tau} \exp\left(\frac{\varkappa}{2} \ln \tau \right) = \text{const} \cdot \sqrt{\tau}\, \tau^{\varkappa/2}.$$

Hence, $\varkappa$ must be 1 or $3, 5, \ldots$. Since $\Gamma_3^{(1,2)}$ is an elliptic function of the second kind, the total number of its zeros on E must be equal to that of its poles (with multiplicity), that is, k, therefore the residuum $\varkappa$ must be 1. The same argumentation for $\alpha = 1, 2$ completes the proof. $\square$

5 Sigma-function solutions of 2nd kind

In order to convert the algebraic solutions of Theorem 4 to analytic ones, we shall need the following formula.

Proposition 5. *Let $K_k(\lambda)$ be a polynomial of odd degree k, and*

$$W = \frac{K_k(\lambda)}{Q_k(\lambda)} \frac{\mathrm{d}\lambda}{2\sqrt{U_4(\lambda)}}, \qquad Q_k(\lambda) = r_0(\lambda - r_1)\cdots(\lambda - r_k)$$

be a differential of the third kind on the degree 4 curve E with simple poles at the points $\mathcal{P}_j^{\pm} = (r_j, \pm\sqrt{U_4(r_j)})$, $j = 1, \ldots, k$ with residues ± 1 respectively. Let the point $(\lambda, \mu) \in E$ and $u \in \mathbb{C}$ be related by the Abel map (3.8). Then

$$\int_{(c,0)}^{(\lambda,\mu)} W = \log \frac{\sigma(u - w_1)\cdots\sigma(u - w_k)}{\sigma(u + w_1)\cdots\sigma(u + w_k)} + 2[\zeta(w_1) + \cdots + \zeta(w_k)]u + \delta\, u - \pi\,\mathrm{i}, \tag{5.1}$$

$$\frac{\sqrt{(\lambda - r_1)\cdots(\lambda - r_k)}}{\sqrt{(\lambda - c)^k}} = \mathrm{const}\frac{\sqrt{\sigma(u - w_1)\cdots\sigma(u - w_k)\,\sigma(u + w_1)\cdots\sigma(u + w_k)}}{\sigma^k(u)}, \tag{5.2}$$

where, as above, $\zeta(u)$ is the Weierstrass zeta-function, $\delta = K(c)/Q(c)$, and

$$w_j = \mathrm{i}\int_{(c,0)}^{\mathcal{P}_j^-} \frac{\mathrm{d}\lambda}{2\sqrt{U_4(\lambda)}}\,. \tag{5.3}$$

The correct signs of the roots w_j can be chosen from the conditions

$$\frac{K_k(r_j)}{(r_j - c)Q_k'(r_j)} = -\frac{3\mathrm{i}}{(a_1 - c)(a_2 - c)(a_3 - c)}\wp'(w_j). \tag{5.4}$$

The proposition is a reformulation of known relations in the theory of elliptic functions, its proof is purely technical and given in Appendix 2.

Note that if the polynomial $K_k(\lambda)$ contains the factor $(\lambda - c)^s$, $s \geq 1$, the constant δ in (9.12) is zero.

Theorem 4 and Proposition 5 allow us to formulate the following theorem.

Theorem 6. *1) The two complex vector elliptic solutions of the second kind of the Poisson equations are*

$$G^{(1)}(u) = \left(G_1^{(1)}, G_2^{(1)}, G_3^{(1)}\right)^T, \qquad G^{(2)}(u) = \left(G_1^{(2)}, G_2^{(2)}, G_3^{(2)}\right)^T, \tag{5.5}$$

$$G_\alpha^{(1)}(u) = \epsilon_\alpha \mathrm{e}^{\Theta_\alpha u} \prod_{l=1}^{k} \frac{\sigma(u - w_{l,\alpha})}{\sigma(u)\,\sigma(-w_{l,\alpha})}, \qquad G_\alpha^{(2)}(u) = \epsilon_\alpha \mathrm{e}^{-\Theta_\alpha u} \prod_{l=1}^{k} \frac{\sigma(u + w_{l,\alpha})}{\sigma(u)\,\sigma(w_{l,\alpha})},$$

$$w_{l,\alpha} = \mathrm{i}\int_c^{r_{l,\alpha}} \frac{\mathrm{d}\lambda}{2\sqrt{U_4(\lambda)}}\,, \qquad l = 1, \ldots, k, \quad \alpha = 1, 2, 3,$$

where $r_{l,\alpha}$ *are the roots of the polynomials* $Q_\alpha(\lambda)$ *in* (4.3)*, the signs of* $w_{j,i}$ *are defined according to* (5.3)*,* (5.4)*. Next,* ϵ_α *are specified in* (3.21)*,* $u = \mathrm{i}mt + \Omega_2$*, and*

$$\Theta_1 = \sum_{j=1}^{k} \zeta(w_{j,1}), \quad \Theta_2 = \sum_{j=1}^{k} \zeta(w_{j,2}), \quad \Theta_3 = \sum_{j=1}^{k} \zeta(w_{j,3}).$$

Together with (3.20)*,* (3.21)*, the expressions* (5.5) *form a complete basis of independent solutions of the equations* (1.11)*.*

2) Let also

$$\Sigma_1 = \sum_{j=1}^{k} w_{j,1}, \quad \Sigma_2 = \sum_{j=1}^{k} w_{j,2}, \quad \Sigma_3 = \sum_{j=1}^{k} w_{j,3}.$$

The solutions (5.5) *have the quasi-monodromy*

$$G_\alpha^{(1)}(u+2\Omega_j) = (-1)^{\delta_{\alpha j}} s_j\, G_\alpha^{(1)}(u), \quad G_\alpha^{(2)}(u+2\Omega_j) = (-1)^{\delta_{\alpha j}} s_j^{-1}\, G_\alpha^{(2)}(u), \qquad (5.6)$$
$$j = 1,2,3, \quad \alpha = 1,2,3$$

and imply the vector monodromy

$$G^{(1)}(u+4\Omega_j) = s_j^2 G^{(1)}(u), \quad G^{(2)}(u+4\Omega_j) = s_j^{-2} G^{(2)}(u), \qquad (5.7)$$

where

$$\begin{aligned} s_1 &= -\exp(2\Theta_1\Omega_1 - 2\Sigma_1\eta_1) = \exp(2\Theta_2\Omega_1 - 2\Sigma_2\eta_1) = \exp(2\Theta_3\Omega_1 - 2\Sigma_3\eta_1), \\ s_2 &= -\exp(2\Theta_2\Omega_2 - 2\Sigma_2\eta_2) = \exp(2\Theta_1\Omega_2 - 2\Sigma_1\eta_2) = \exp(2\Theta_3\Omega_2 - 2\Sigma_3\eta_2), \\ s_3 &= -\exp(2\Theta_3\Omega_3 - 2\Sigma_3\eta_3) = \exp(2\Theta_2\Omega_3 - 2\Sigma_2\eta_3) = \exp(2\Theta_1\Omega_3 - 2\Sigma_1\eta_3), \end{aligned} \qquad (5.8)$$

3) If Ω_j *is the imaginary half-period, then* $|s_j| = 1$*. For the real half-period* Ω_2 *one has* $|s_2| \neq 1$*. Moreover,*

$$\Sigma_\alpha - \Sigma_\beta = \Omega_\gamma \mod \{2\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\}, \quad \Theta_\alpha - \Theta_\beta = \eta_\gamma \mod \{2\eta_1\mathbb{Z} + 2\eta_2\mathbb{Z}\}. \qquad (5.9)$$

for $(\alpha, \beta, \gamma) = (1,2,3)$*.*

4) Finally, for any $u \in \mathbb{C}$*,*

$$\langle G^{(1)}(u), G^{(1)}(u)\rangle = 0, \qquad \langle G^{(2)}(u), G^{(2)}(u)\rangle = 0, \qquad (5.10)$$

$$G_\alpha^{(1)}(u) G_\alpha^{(2)}(u) = -\epsilon_\alpha^2 \prod_{r=1}^{k} (\wp(u) - \wp(w_{r,\alpha})), \qquad (5.11)$$

and

$$\begin{aligned} \Big[G_\alpha^{(1)}(\Omega_j)\Big]^2 &= s_j\ (-1)^{1-\delta_{\alpha j}} \epsilon_i^2 \prod_{r=1}^{k} (\wp(\Omega_j) - \wp(w_{r,\alpha})), \\ \Big[G_\alpha^{(2)}(\Omega_j)\Big]^2 &= s_j^{-1} (-1)^{1-\delta_{\alpha j}} \epsilon_i^2 \prod_{r=1}^{k} (\wp(\Omega_j) - \wp(w_{r,\alpha})), \end{aligned} \qquad (5.12)$$
$$\alpha, j = 1,2,3.$$

Moreover, for any $u \in \mathbb{C}$,

$$
\begin{aligned}
-\langle G^{(1)}(u), G^{(2)}(u)\rangle &= \epsilon_1^2 \prod_{l=1}^{k} (\wp(\Omega_1) - \wp(w_{l,1})) \\
= \epsilon_2^2 \prod_{l=1}^{k} (\wp(\Omega_2) - \wp(w_{l,2})) &= \epsilon_3^2 \prod_{l=1}^{k} (\wp(\Omega_3) - \wp(w_{l,3})) = -\Pi,
\end{aligned} \tag{5.13}
$$

where the constant Π *is defined in* (3.23), (3.24).

The proof of the Theorem can be found in Appendix 2.

Remark. One can easily recognize that, for each index α, the components $G_\alpha^{(1)}(u), G_\alpha^{(2)}(u)$ have the same structure as solutions of the Lame equation

$$
\frac{d^2\Lambda}{d\, u^2} = (n(n+1)\wp(u) + B)\Lambda, \quad n \in \mathbb{N}, \quad B = \text{const}
$$

with $n = k$ (see, e.g., [12]), namely,

$$
\Lambda_1 = \prod_{s=1}^{k} \left(\frac{\sigma(u - h_s)}{\sigma(u)\,\sigma(h_s)} \exp(\zeta(h_s)u) \right), \quad \Lambda_2 = \prod_{s=1}^{k} \left(\frac{\sigma(u + h_s)}{\sigma(u)\,\sigma(h_s)} \exp(-\zeta(h_s)u) \right),
$$

where the zeros $h_1, \dots, h_k$ satisfy various conditions, in particular,

$$
\wp(h_1) + \cdots + \wp(h_k) = kB.
$$

However, as numerical tests show, the zeros $w_{1,\alpha}, \dots, w_{k,\alpha}$ of the solutions (5.5) do not satisfy all the conditions on $h_1, \dots, h_k$. Hence $G_\alpha^{(1)}(u), G_\alpha^{(2)}(u)$ cannot be solutions of the Lame equation.

Thus, if the relation between the Poisson equations (1.11) and the Lame equation (or some of its generalizations) exists, it should be a rather non-trivial one.

6 The classical case $k = 1$

The Poisson equations in this case were first integrated by C. Jacobi [4], who used previous results of Legendre (see, e.g., [11]). The case does not fit completely into Theorems 4, 6 because the corresponding meromorphic differentials (4.2) do not contain the factor $(\lambda - c)^s$, and the solutions do not have precisely the structure of (5.5).

Namely, now the elliptic solution P is just $M(u)$ given by (3.20) and the algebraic solutions (4.1) reread

$$
G_\alpha^{(1,2)} = c_\alpha \sqrt{M_\beta^2 + M_\gamma^2} \exp\left(\pm \frac{1}{2} \int W_\alpha \right), \qquad (\alpha, \beta, \gamma) = (1, 2, 3).
$$

Set, for concreteness, $\alpha = 3$. Using the algebraic parameterization (3.3) for $M(\lambda)$, from (4.7) we get

$$
\begin{aligned}
W_3 &= 2\mathrm{i}\frac{a_1 M_1^2(\lambda) + a_2 M_2^2(\lambda)}{M_1^2(\lambda) + M_2^2(\lambda)} dt \\
&= 2\frac{(ca_3 - a_1a_2)\lambda + c(a_1a_2 - a_1a_3 - a_2a_3) - a_1a_2a_3}{(c + a_3 - a_1 - a_2)\lambda - (ca_3 - a_1a_2)} \, \frac{d\lambda}{2\sqrt{U_4(\lambda)}}.
\end{aligned} \tag{6.1}
$$

This is a differential of 3rd kind having a pair of simple poles $(\lambda^*, \pm\sqrt{U_4(\lambda^*)})$ on E with

$$
\lambda^* = \frac{a_1a_2 - ca_3}{c + a_3 - a_1 - a_2}.
$$

Observe that in (6.1)

$$
2\frac{(ca_3 - a_1a_2)\lambda + c(a_1a_2 - a_1a_3 - a_2a_3) - a_1a_2a_3}{(c + a_3 - a_1 - a_2)\lambda - (ca_3 - a_1a_2)}\bigg|_{\lambda=c} = 2a_3.
$$

Then, according to Proposition 5,

$$
\begin{aligned}
\int_{(c,0)}^{(\lambda,\mu)} W_3 &= \log\frac{\sigma(u - w_3)}{\sigma(u + w_3)} + 2[\zeta(w_3) + a_3]u - \pi\mathrm{i}, \\
\sqrt{M_1^2 + M_2^2} &= \text{const}\, \frac{\sqrt{\sigma(u - w_3)\,\sigma(u + w_3)}}{\sigma(u)},
\end{aligned}
$$

where w_3 is the Abel image of the pole (λ^*, μ) of W_3 with residuum -1,

$$
w_3 = \int_{(c,0)}^{\lambda^*} \frac{\mathrm{i}\, d\lambda}{2\sqrt{U_4(\lambda)}}.
$$

As a result, up to multiplication by -1, the elliptic solutions of 2nd kind are

$$
\begin{aligned}
G_\alpha^{(1)}(u) &= \epsilon_\alpha \frac{\sigma(u - w_\alpha)}{\sigma(u)\,\sigma(-w_\alpha)} \exp\left[(\zeta(w_\alpha) + a_\alpha)u\right], \\
G_\alpha^{(2)}(u) &= \epsilon_\alpha \frac{\sigma(u - w_\alpha)}{\sigma(u)\,\sigma(-w_\alpha)} \exp\left[-(\zeta(w_\alpha) + a_\alpha)u\right]
\end{aligned} \tag{6.2}
$$

(compare with (5.5)), where w_α denote the Abel image of the pole of the differential W_α with the residuum -1, and, as above, $u = \mathrm{i}mt + \Omega_2$.

As follows from item 3 of Theorem 6, here

$$
\begin{gathered}
w_\alpha - w_\beta = \Omega_\gamma \mod \{2\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\}, \\
\zeta(w_\alpha) - \zeta(w_\beta) = \eta_\gamma = \zeta(\Omega_\gamma) \mod \{2\eta_1\mathbb{Z} + 2\eta_2\mathbb{Z}\}, \\
(\alpha, \beta, \gamma) = (1, 2, 3).
\end{gathered}
$$

Then, introducing

$$
w^* = w_\alpha - \Omega_\alpha, \quad \theta^* = \zeta(w_\alpha) - \eta_\alpha = \zeta(w^*) \quad \text{for any } \alpha
$$

and using the definition (3.13) of the sigma-functions with indices, one can represent the complex solutions (6.2) in the following form

$$
\begin{aligned}
G_\alpha^{(1)}(u) &= \epsilon_\alpha \frac{\sigma_\alpha(u-w^*)}{\sigma(u)\,\sigma_\alpha(-w^*)} \exp\left[(\theta^*+a_\alpha)u\right], \\
G_\alpha^{(2)}(u) &= \epsilon_\alpha \frac{\sigma_\alpha(u+w^*)}{\sigma(u)\,\sigma_\alpha(w^*)} \exp\left[-(\theta^*+a_\alpha)u\right].
\end{aligned} \tag{6.3}
$$

By transforming the integrals defining w_α, one can show that

$$
w^* = \int_{(c,0)}^{\infty} \frac{\mathrm{i}\, d\lambda}{2\sqrt{U_4(\lambda)}},
$$

where ∞ stands for one of the two infinite points on E.

Finally notice that being rewritten in terms of theta-functions, the expressions (6.3) coincide with the complex solutions presented by C. Jacobi (see also [11]).

7 Real normalized vector solutions.

As was shown in Section 2, the elliptic solution $P(u)$ in (3.20), (3.21) for $u = \mathrm{i}nt + \Omega_2$, $t \in \mathbb{R}$, is real. Then, by their construction (see (4.6)), the basis elliptic 2nd kind solutions $G_i^{(1)}(u), G_i^{(2)}(u)$ have opposite arguments:

$$
\operatorname{Arg}\left(G_\alpha^{(1)}(u)\right) = -\operatorname{Arg}\left(G_\alpha^{(2)}(u)\right) \tag{7.1}
$$

for any $\alpha = 1, 2, 3$. Then two independent *non-normalized* real vector solutions can be written in the form

$$
\begin{aligned}
\gamma^{(1)}(t) &= \nu_1 G^{(1)}(\mathrm{i}mt+\Omega_2) + \nu_2 G^{(2)}(\mathrm{i}mt+\Omega_2), \\
\gamma^{(2)}(t) &= \frac{1}{\mathrm{i}}\left[\nu_1 G^{(1)}(\mathrm{i}mt+\Omega_2) - \nu_2 G^{(2)}(\mathrm{i}mt+\Omega_2)\right],
\end{aligned} \tag{7.2}
$$

with some appropriate constants ν_1, ν_2. In view of (7.1), it is sufficient to set

$$
\nu_1 = \chi |G_\alpha^{(1)}(\mathrm{i}mt^*+\Omega_2)|^{-1}, \quad \nu_2 = \chi |G_\alpha^{(2)}(\mathrm{i}mt^*+\Omega_2)|^{-1}, \tag{7.3}
$$

for any fixed real t^*, any real nonzero χ, and any $\alpha \in \{1, 2, 3\}$. Then we arrive at

Theorem 7. *A real orthogonal rotation matrix formed by the three independent unit vector solutions of the Poisson equations* (1.11) *has the form*

$$
\mathcal{R}(t) = \frac{1}{\sqrt{\Pi}}\left(\bar\gamma^{(1)}(t), \bar\gamma^{(2)}(t), \bar P(t)\right), \qquad \bar P(t) = P(\mathrm{i}mt+\Omega_2), \tag{7.4}
$$

$$
\bar\gamma^{(1)}(t) = \frac{1}{2}\left[\frac{1}{\sqrt{s_2}} G^{(1)}(\mathrm{i}mt+\Omega_2) + \sqrt{s_2}\, G^{(2)}(\mathrm{i}mt+\Omega_2)\right],
$$

$$
\bar\gamma^{(2)}(t) = \frac{1}{2}\left[\frac{1}{\sqrt{-s_2}} G^{(1)}(\mathrm{i}mt+\Omega_2) + \sqrt{-s_2}\, G^{(2)}(\mathrm{i}mt+\Omega_2)\right],
$$

where $P(u)$ is the elliptic solution (3.20), (3.21), *and $G^{(1)}(u), G^{(2)}(u)$ are the elliptic solutions of the second kind described in* (5.5) *and Theorem 6. Next, s_2 is real and is specified in* (5.8), *whereas the constant Π is defined in* (3.23), (3.24).

Note that the columns of $\mathcal{R}(t)$ form a left- or right-oriented orthonormal basis.

Proof of Theorem 7. Setting in (7.3) $t^*=0, \alpha=2$, and

$$\chi=\frac{\epsilon_2}{2}\sqrt{(\wp(\Omega_2)-\wp(\alpha_{1,2}))\cdots(\wp(\Omega_2)-\wp(\alpha_{k,2}))}\,,$$

in view of (5.12), we get $\nu_1=s_2^{-1/2}, \nu_2=s_2^{1/2}$. Then (7.2) gives

$$\gamma^{(1)}(0)=\frac{1}{2}\left[\frac{1}{\sqrt{s_2}}G^{(1)}(\Omega_2)+\sqrt{s_2}G^{(2)}(\Omega_2)\right],$$
$$\gamma^{(2)}(0)=\frac{1}{2}\left[\frac{1}{\sqrt{-s_2}}G^{(1)}(\Omega_2)+\sqrt{-s_2}G^{(2)}(\Omega_2)\right].$$

Then, in view of (5.10) and (5.13)

$$\sum_{\alpha=1}^{3}\left[\gamma_\alpha^{(1)}(0)\right]^2=\sum_{\alpha=1}^{3}\left[\gamma_\alpha^{(2)}(0)\right]^2=-\frac{1}{2}\langle G^{(1)}(\Omega_2),G^{(2)}(\Omega_2)\rangle=\Pi. \tag{7.5}$$

As a result, the real vectors $\bar\gamma^{(1)}(t),\bar\gamma^{(2)}(t),\bar P(t)$ all have the same length. By their construction, they are all orthogonal. Hence, we obtain the matrix $\mathcal{R}(t)$ in (7.4). □

8 The case of negative odd k.

We first note that the case of negative odd k cannot be reduced to the already considered case $k>0$ by the trivial substitution $t=-T$ in the Poisson equations in (1.6). Indeed, this change gives

$$\frac{d\gamma}{dT}=-k\gamma\times aM(-T)$$

The elliptic vector solution $M(-T)$ given by (3.20) with $u=-\mathrm{i}mT+\Omega_2$ is neither odd nor even, hence one cannot write $M(-T)=M(T)$, and the above equation cannot be transformed to the form

$$\frac{d\gamma}{dT}=-k\gamma\times aM(T).$$

Nevertheless, the analysis for positive k is sufficient to cover all the cases. Indeed, upon introducing new moments of inertia

$$\Lambda_i=(J_j+J_k-J_i)J_i,\quad (i,j,k)=(1,2,3), \tag{8.1}$$

the system (1.6) can be rewritten as

$$\dot{\Lambda\omega}=-\Lambda\omega\times\omega,\quad \dot\gamma=k\gamma\times\omega, \tag{8.2}$$

which, under the change $t=-T$, gives

$$\Lambda\omega'=\Lambda\omega\times\omega,\quad \gamma'=-k\gamma\times\omega,\qquad (')=\frac{d}{dT}. \tag{8.3}$$

The Euler equations here have the integrals

$$\langle\omega,\Lambda\omega\rangle = L, \quad \langle\omega,\Lambda^2\omega\rangle = \mathcal{M}^2$$

with integration constants $L, \mathcal{M}$. Then, applying to these equations the procedure of section 3, we express the solutions $\omega(T)$ in terms elliptic functions of the curve

$$E' = \{W^2 = -(Z-A_1)(Z-A_2)(Z-A_3)(Z-C)\}$$

with the parameters

$$\begin{gathered} A_i = 1/\Lambda_i = \frac{a_i^2 a_j a_k}{a_i a_j + a_i a_k - a_j a_k}, \\ C = \frac{L}{\mathcal{M}^2} = \frac{(\tau_2\, c - 2\,\tau_3)\,\tau_3}{\tau_2^2 - 4\tau_1\tau_3 + 4\,\tau_3\, c}\,, \quad \mathcal{M}^2 = \langle\omega,\Lambda^2\omega\rangle = \frac{\tau_2^2 - 4\tau_1\tau_3 + 4\,\tau_3\, c}{\tau_3^2} m^2, \\ \tau_1 = a_1 + a_2 + a_3, \qquad \tau_2 = a_1a_2 + a_2a_3 + a_3a_1, \qquad \tau_3 = a_1a_2a_3. \end{gathered}$$

Here, as above, in (1.3),

$$a_i = 1/J_i, \quad c = l/m^2, \quad m^2 = \langle M, M\rangle = \langle\omega, J^2\omega\rangle,$$

Since $\omega(T)$, like $M(t)$, must also be elliptic functions of the original curve E, we get the following relation

$$\begin{aligned} dt &= \frac{d\lambda}{2m\sqrt{-(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)(\lambda-c)}} \\ &= -\frac{dZ}{2\mathcal{M}\sqrt{-(\lambda-A_1)(\lambda-A_2)(\lambda-A_3)(\lambda-C)}}. \end{aligned} \tag{8.4}$$

Remark. As one may expect, the elliptic curves E, E' with the parameters a_i, c and A_i, C are birationally equivalent. Indeed, E' is transformed to E by the substitution

$$\begin{aligned} Z &= \frac{(\tau_2\,\lambda - 2\,\tau_3)\,\tau_3}{\tau_2^2 - 4\tau_1\tau_3 + 4\,\tau_3\,\lambda}, \\ W &= \mu \frac{(a_1a_2 - a_2a_3 - a_1a_3)(a_1a_3 - a_2a_3 - a_2a_3)(a_2a_3 - a_1a_3 - a_1a_2)}{(\tau_2^2 - 4\tau_1\tau_3 + 4\,\tau_3\,\lambda)^2\,(\tau_2^2 - 4\tau_1\tau_3 + 4\,\tau_3\, c)}. \end{aligned}$$

We stress that the half-periods of E and E'

$$\Omega_\alpha = \int_c^{a_\alpha} \frac{\mathrm{i}d\lambda}{2\sqrt{U_4(\lambda)}}, \quad \Omega'_\alpha = \int_C^{A_\alpha} \frac{\mathrm{i}dZ}{2\sqrt{-(\lambda-A_1)(\lambda-A_2)(\lambda-A_3)(\lambda-C)}}, \\ \alpha = 1,2,3$$

in general, do not coincide, but only proportional to each other: in view of (8.4), $\Omega'_\alpha = \frac{\mathcal{M}}{m}\Omega_\alpha$, $\alpha = 1,2,3$.

Now comparing (1.6) or (8.2) with (8.3), we arrive at the following observation.

Proposition 8. *Let k be an odd negative integer and the vector $\gamma(T) = \gamma(T|A_1, A_2, A_3, C, \mathcal{M})$ be a solution of the Poisson equations in* (8.3) *with the elliptic coefficients $\omega(T)$ related to the parameters $A_i, C, \mathcal{M}$. Then $\gamma(t) = \gamma(-T)$ is a solution of the Poisson equations* (1.11) *with the elliptic oefficients $M_\alpha(t)$ related to the parameters a_i, c, m, and vice versa.*

In other words, the solutions $\gamma(t)$ of (1.11) with an odd negative k and the parameters a_1, a_2, a_3, c, m are given by $\gamma(-t|A_1, A_2, A_3, C, \mathcal{M})$. The latter are described by the formulae of Theorems 4 and 6 corresponding to $|k|$, the parameters $A_1, A_2, A_3, C, \mathcal{M}$, and the corresponding roots of the polynomials $F_{s,\alpha}(\lambda), Q_{|k|,\alpha}(\lambda)$.

We stress that although the elliptic vector functions $\omega(t)$ for a_1, a_2, a_3, c, m and $\omega(-t)$ for $A_1, A_2, A_3, C, \mathcal{M}$, coincide, this is no more true for the solutions $\gamma(t|a_1, a_2, a_3, c, m)$ and $\gamma(-t|A_1, A_2, A_3, C, \mathcal{M})$.

9 Comparison with the Halphen equation

By using the algebraic parametrization (3.3), the generalized Poisson equations (1.11) can be rewritten as 3rd order ODE for one of the components of the vector γ, say γ_1, with the independent variable $\lambda \in \mathbb{C}$. For general a_i, k the explicit expressions for the coefficients of the ODE are very long. We give an example for $a_1 = 1, a_2 = 2, a_3 = 3$ and $k = 3$ (c is arbitrary):

$$\frac{d^3}{d\lambda^3}\gamma_1 + g_2(\lambda)\frac{d^2}{d\lambda^2}\gamma_1 + g_1(\lambda)\frac{d}{d\lambda}\gamma_1 + g_0(\lambda)\gamma_1 = 0, \tag{9.1}$$

with

$$\begin{aligned}
g_2 &= \frac{3}{2\,(\lambda-2)} + \frac{3}{2\,(\lambda-3)} + \frac{3}{\lambda - c} - \frac{c+10}{10\,\lambda - 15\,c + 6 + c\,\lambda} + \frac{1}{\lambda-1},\\
g_1 &= -\frac{3\,(138\,c-37)}{52\,(c-2)\,(\lambda-2)} + \frac{82\,c-119}{24\,(c-3)\,(\lambda-3)} - \frac{34\,c^2-276\,c+263}{4\,(c-2)\,(c-3)\,(c-1)\,(c-\lambda)}\\
&\quad - \frac{1}{4\,(\lambda-1)^2} + \frac{(10+c)^2\,(10\,c^2+2267\,c-2666)}{156\,(c-2)\,(-3+c)\,(7\,c-8)\,(-15\,c+10\,\lambda+6+c\,\lambda)}\\
&\quad - \frac{5}{4\,(c-\lambda)^2} + \frac{254\,c^2-669\,c+436}{8\,(7\,c-8)\,(c-1)\,(\lambda-1)},\\
g_0 &= -\frac{27\,(c-1)\,(23\,c-30)}{104\,(\lambda-2)\,(c-2)^2} + \frac{3\,(c-1)\,(17\,c-22)}{32\,(\lambda-3)\,(c-3)^2} + \frac{9\,(4\,c^3-81\,c^2+171\,c-98)}{8\,(c-\lambda)\,(c-1)^2\,(c-2)^2\,(c-3)^2}\\
&\quad + \frac{9\,(5\,c-6)}{16\,(c-1)\,(-1+\lambda)^2} - \frac{3\,(-1+c)\,(10+c)^5}{208\,(c-2)^2\,(-3+c)^2\,(7\,c-8)\,(-15\,c+10\,\lambda+6+c\,\lambda)}\\
&\quad + \frac{9}{8\,(c-3)\,(c-1)\,(c-\lambda)^2} + \frac{9\,(109\,c^3-235\,c^2+106\,c+24)}{32\,(-1+c)^2\,(7\,c-8)\,(\lambda-1)}.
\end{aligned}$$

That is, the coefficients have poles at $\lambda = a_1, a_2, a_3, c$, and at an extra pole defined by the condition $-15\,c + 6 + (10+c)\,\lambda = 0$.

A natural question is how the above 3rd order equation is related with known linear equations with elliptic coefficients admitting elliptic solutions of second kind. The best known example is the *Halphen equation*

$$\frac{d^3}{du^3}\Psi + (1-n^2)\wp(u)\frac{d}{du}\Psi + \wp'(u)\frac{1-n^2}{2}\Psi = h\Psi, \qquad \Psi = \Psi(u), \tag{9.2}$$

where n is integer and h is an arbitrary parameter. As above, $\wp(u)$ is the Weierstrass function. For any such n the 3 independent solutions $\Psi_1(u), \Psi(u), \Psi_3(u)$ are elliptic functions of 2nd kind with poles of order $g = n-1$ at $u = 0$:

$$\Psi_\alpha(u) = \frac{\sigma(u - w_1^{(\alpha)}(h))\cdots\sigma(u - w_g^{(\alpha)}(h))}{\sigma^g(u)} \exp[\,(\zeta(w_1^{(\alpha)}) + \cdots + \zeta(w_g^{(\alpha)}))u], \qquad \alpha = 1,2,3.$$

The structure of the solutions generalizes that of solutions (5.5) of our equation (9.1). So, one can suppose that the equation (9.1) is a special case of the Halphen equation (for $h = 0$), when one of its solution is elliptic.

However, written in the algebraic form with the independent variable z such that

$$u = \int_z^\infty \frac{dz}{2\sqrt{(Z-e_1)(Z-e_2)(Z-e_3)}}, \quad \wp(u) = z$$

the Halphen equation with $h = 0$ is

$$4(z-e_1)(z-e_2)(z-e_3)\frac{d^3}{dz^3}\Psi + g_2(z)\frac{d^2}{dz^2}\Psi + g_1(z)\frac{d}{dz}\gamma_1 + g_0(z)\Psi = 0,$$
$$g_2 = 36z^2 + 6e_1e_2 + 6e_1e_3 + 6e_2e_3, \quad g_1 = 12z - 1 - n^2, \quad g_0 = -(1-n^2)/2.$$

Hence, the coefficients of the normalized equation have finite poles only at $z = e_1, e_2, e_3$. Taking into account that the equation (9.1) has 5 poles (which can be reduced to 4 finite poles), it cannot be identified with the special case of the Halphen equation.

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Appendix 1: A numerical example.

This example was made with `Maple` by using the functions `WeierstrassP(u,g[2],g[3])`, `WeierstrassSigma(u,g[2],g[3])`, `WeierstrassZeta(u,g[2],g[3])`. Consider the simplest nontrivial case $k = 3$ and $B = ka$. Choose the inertia tensor $J = \mathrm{diag}(1, 1/2, 1/3)$, i.e., $a = \mathrm{diag}(1,2,3)$ and $l/n^2 = c = 5/2 \in [a_2, a_3]$, $n = 1$. Therefore, in the real case the parameter $\lambda \in [a_1, a_2] = [1; 2]$. The elliptic curve E has the form

$$\mu^2 = 2U_4(\lambda) = -2(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-5/2).$$

The birational transformation (3.5), namely,

$$z = -\frac{1}{12}\frac{\lambda+2}{z-5/2}, \qquad \lambda = 2\frac{15z-1}{12z+1}$$

takes it to the Weierstrass form

$$w^2 = 4z^3 - \frac{7}{3}z + \frac{10}{27} = 4(z+5/6)(z-1/6)(z-2/3),$$
$$\text{with} \qquad z(\lambda=1)=1/6, \quad z(2)=2/3, \quad z(3)=-5/6.$$

so that the parameters of the Weierstrass functions of E are $g_2 = 7/3$, $g_3 = -10/27$. The half-periods are[3]:

$$\Omega_1 = \mathrm{i}\int_c^{a_1}\frac{d\lambda}{\mu} = \int_\infty^{1/6}\frac{dz}{w} = -1.656638 - 1.415737\cdot \mathrm{i},$$
$$\Omega_2 = \int_\infty^{2/3}\frac{dz}{w} = 1.656638, \quad \Omega_3 = \int_\infty^{-5/6}\frac{dz}{w} = 1.415737\cdot \mathrm{i}.$$

The corresponding constants $\eta_i = \zeta(\Omega_i)$ in (3.13) are

$$\eta_1 = -0.4402056 + 0.57199\cdot \mathrm{i}, \quad \eta_2 = 0.4402056, \quad \eta_3 = -0.57199\cdot \mathrm{i}.$$

This allows to calculate

$$\sigma_\alpha(u) = \exp(\eta_\alpha u)\,\frac{\sigma(\Omega_\alpha - u|g_2,g_3)}{\sigma(\Omega_\alpha|g_2,g_3)}, \qquad \alpha = 1,2,3.$$

Next,

$$\epsilon_1 = \frac{1}{\sqrt{(a_1-a_2)(a_1-a_3)}} = \frac{\sqrt{2}}{2}, \quad \epsilon_2 = -\mathrm{i}, \quad \epsilon_3 = \frac{\sqrt{2}}{2}.$$

From (3.18) we have

$$F_{11}(\lambda) = -34\lambda + 167/2, \quad F_{12} = -48\lambda + 237/2, \quad F_{13} = -66\lambda + 327/2,$$

The Abel images of their zeros on the complex plane u are

$$v_1 = \pm 0.34497195, \quad v_2 = \pm 0.2898, \quad v_3 = \pm 0.24686\,.$$

Then the *real* elliptic solutions for the Euler and the Poisson equations are given by

$$M_\alpha = \epsilon_\alpha\frac{\sigma_\alpha(u)}{\sigma(u)}, \quad P_\alpha = \epsilon_\alpha\frac{\sigma_\alpha(u)\sigma(u-v_\alpha)\sigma(u+v_\alpha)}{\sigma^3(u)}, \tag{9.3}$$
$$\alpha = 1,2,3, \quad u = \mathrm{i}t + \Omega_2, \qquad t\in\mathbb{R},$$

with the indicated above values of the parameters. Note that here $|M| = 1$ and $|P|^2 = \Pi = 201.062507$.

[3]In this example most of the float numbers are indicated up to 10^{-6}.

Next, the meromorphic differentials in (4.2) are

$$W_1 = \mathrm{i}\,\frac{3\sqrt{3217}\,(2y-5)(260\,y^2-1295\,y+1613)}{2(3984\,y^3-29608\,y^2+73363\,y-60607)\sqrt{U_4(\lambda)}},$$
$$W_2 = \mathrm{i}\,\frac{3\sqrt{3217}\,(2\,y-5)(560\,y^2-3136\,y+4331)}{2(23824\,y^3-194232\,y^2+523569\,y-467236)\sqrt{U_4(\lambda)}},$$
$$W_3 = \mathrm{i}\,\frac{3\sqrt{3217}\,(2\,y-5)(1084\,y^2-4913\,y+5521)}{2(50672\,y^3-356280\,y^2+832461\,y-646139)\sqrt{U_4(\lambda)}}.$$

Then, up to constant factors, the polynomials $Q_{3,\alpha}$ in Theorem 4 are

$$Q_{3,3}(\lambda) = 3167/2\,\lambda^3 - 44535/4\,\lambda^2 + 832461/32\,\lambda - 646139/32,$$
$$Q_{3,2}(\lambda) = 1489\,\lambda^3 - 24279/2\,\lambda^2 + 523569/16\,\lambda - 116809/4,$$
$$Q_{3,1}(\lambda) = 6723/2\,\lambda^3 - 99927/4\,\lambda^2 + 1980801/32\,\lambda - 1636389/32\,.$$

The Abel images of their zeros on the complex u-plane are

$$w_{1,3} = \pm 0.21963966, \quad w_{2,3} = \pm 0.309571742, \quad w_{3,3} = \pm 1.232049297,$$
$$w_{1,2} = \pm(\Omega_3 + 0.40885), \quad w_{2,2}, w_{3,2} = \pm(0.272475 \pm 0.041941\cdot\mathrm{i}),$$
$$w_{1,1} = -0.26888528, \quad w_{2,1}, w_{3,1} = \pm(0.3424574955 \pm 0.2549408658\cdot\mathrm{i}).$$

Applying the condition (5.4), we choose the signs

$$\begin{aligned}
&w_{1,3} = -0.21963966, \quad w_{2,3} = -0.309571742, \quad w_{3,3} = 1.232049297,\\
&w_{1,2} = -0.40885 + \Omega_3, \quad w_{2,2} = -0.272475 + \Omega_3,\\
&w_{3,2} = -0.272475 - \Omega_3,\\
&w_{1,1} = -0.26888528, \quad w_{2,1} = -0.3424574955 - 0.2549408658\cdot\mathrm{i},\\
&w_{3,1} = -0.3424574955 + 0.2549408658\cdot\mathrm{i}.
\end{aligned} \tag{9.4}$$

Then

$$\Sigma_3 = \sum_{j=1}^{3} w_{j,3} = 0.7028378946, \quad \Sigma_2 = -0.95380028 + \Omega_3, \quad \Sigma_1 = -0.95380028$$

and

$$\Theta_3 = \sum_{j=1}^{3} \zeta(w_{j,3}) = -7.03775, \quad \Theta_2 = -7.477955 - 0.572\cdot\mathrm{i}, \quad \Theta_1 = -7.477955\,.$$

Notice that

$$\Sigma_3 - \Sigma_2 = 1.656638 - 1.41573\cdot\mathrm{i} = \tilde{\Omega}_1 \equiv \Omega_1, \quad \Sigma_2 - \Sigma_1 = \Omega_3, \quad \Sigma_3 - \Sigma_1 = \Omega_2 \tag{9.5}$$

and

$$\Theta_3 - \Theta_2 = \zeta(\tilde{\Omega}_1), \quad \Theta_2 - \Theta_1 = \zeta(\Omega_3) = \eta_3, \quad \Theta_3 - \Theta_1 = \zeta(\Omega_2) = \eta_2. \tag{9.6}$$

That is, the sums of zeros of $Q_1(\lambda), Q_2(\lambda), Q_3(\lambda)$ on the complex u-plane differ by the half-periods of the curve E, as predicted by item 3 of Theorem 6. Note that for arbitrary values of $w_{i,j}$, the relations (9.5) does not imply (9.6).

The basis complex vector elliptic solutions of the 2nd kind in (5.5) are

$$
\begin{aligned}
G_\alpha^{(1)}(u) &= \epsilon_\alpha \frac{\sigma(u-w_{1,\alpha})\,\sigma(u-w_{2,\alpha})\,\sigma(u-w_{3,\alpha})}{\sigma^3(u)\,\sigma(-w_{1,\alpha})\,\sigma(-w_{2,\alpha})\,\sigma(-w_{3,\alpha})} e^{\Theta_\alpha u}, \\
G_\alpha^{(2)}(u) &= \epsilon_\alpha \frac{\sigma(u+w_{1,\alpha})\,\sigma(u+w_{2,\alpha})\,\sigma(u+w_{3,\alpha})}{\sigma^3(u)\,\sigma(w_{1,\alpha})\,\sigma(w_{2,\alpha})\,\sigma(w_{3,\alpha})} e^{-\Theta_\alpha u}, \\
&\qquad \alpha = 1,2,3,
\end{aligned}
$$

with $w_{l,\alpha}$ specified in (9.4). Next,

$$
\begin{pmatrix} G_1^{(1)} \\ G_2^{(1)} \\ G_3^{(1)} \end{pmatrix} (u+2\Omega_3) = s_3 \begin{pmatrix} G_1^{(1)} \\ G_2^{(1)} \\ -G_3^{(1)} \end{pmatrix} (u),
$$

$$
s_3 = \exp(2[\Theta_1\Omega_3 - \Sigma_1\eta_3]) = -0.962799 + 0.2702178 \cdot \mathrm{i}, \quad |s_3| = 1,
$$

$$
\begin{pmatrix} G_1^{(1)} \\ G_2^{(1)} \\ G_3^{(1)} \end{pmatrix} (u+2\Omega_2) = s_2 \begin{pmatrix} G_1^{(1)} \\ -G_2^{(1)} \\ G_3^{(1)} \end{pmatrix} (u), \quad s_2 = \exp(2[\Theta_1\Omega_2 - \Sigma_1\eta_2]) = 0.402144 \cdot 10^{-10}.
$$

The monodromy of the solutions $G_i^{(2)}(u)$ is inverse to the above one.

Finally, the ortogonal matrix of real solutions is

$$
\begin{aligned}
\mathcal{R}(t) &= \frac{1}{\sqrt{\Pi}} \left(\bar\gamma^{(1)}(t)\; \bar\gamma^{(2)}(t)\; \bar P(t) \right) \qquad \bar P(t) = P(\mathrm{i}t + \Omega_2), \\
\bar\gamma^{(1)}(t) &= \frac{1}{2}\left[\frac{1}{\sqrt{s_2}} G^{(1)}(\mathrm{i}t+\Omega_2) + \sqrt{s_2}\, G^{(2)}(\mathrm{i}t+\Omega_2) \right], \\
\bar\gamma^{(2)}(t) &= \frac{1}{2}\left[\frac{1}{\sqrt{-s_2}} G^{(1)}(\mathrm{i}t+\Omega_2) + \sqrt{-s_2}\, G^{(2)}(\mathrm{i}t+\Omega_2) \right],
\end{aligned}
$$

with $\sqrt{s_2} = 0.634148 \cdot 10^{-5}$ and $\Pi = 201.0625$.

Appendix 2: Proofs of Proposition 5 and Theorem 6.

Proposition 5 is a reformulation of known relations of the theory of elliptic functions. Consider first an elliptic curve $\mathcal{E}$ in the canonical Weierstrass form (3.6),

$$
\mathcal{E} = \left\{ w^2 = P_3(z) \equiv 4(z-e_1)(z-e_2)(z-e_3) \right\}, \quad e_1+e_2+e_3 = 0,
$$

a point $P = (z,w) = (z, \sqrt{P_3(z)})$ on it, and the Abel map

$$
u = \int_\infty^P \frac{dz}{2\sqrt{(z-e_1)(z-e_2)(z-e_3)}}\, dz, \tag{9.7}
$$

which gives $z = \wp(u)$.

Now let

$$\bar{W} = \frac{\bar{q}_k(z)}{2\sqrt{P_3(z)}\,\bar{Q}_k(z)}\,dz, \qquad \bar{Q}_k(z) = (z - z_1)\cdots(z - z_k), \tag{9.8}$$

be a meromorphic differential of 3rd kind having pairs of only simple poles at the finite points $\mathcal{P}_i^{\pm} = (z_i, \pm 2\sqrt{R_3(z_i)})$, $i = 1, \dots, k$ with residia ± 1 respectively. Here $q_k(z) = b_k z^k + \cdots + b_0$ is a polynomial of degree *at most* k.

Theorem 9. *If u and $P \in \mathcal{E}$ are related by the map (9.7), then, up to an additive constant,*

$$\int_{\infty}^{P} W = \log \frac{\sigma(u - w_1)\cdots\sigma(u - w_k)}{\sigma(u + w_1)\cdots\sigma(u + w_k)} + 2[\zeta(w_1) + \cdots + \zeta(w_k)]u + \varkappa\, u, \tag{9.9}$$

where

$$\wp(\pm w_i) = z_i, \quad \frac{\bar{q}_k(z_i)}{\bar{Q}_k'(z_i)} = -\wp'(w_i) \tag{9.10}$$

$\zeta(u)$ is the Weierstrass zeta function, $\wp'(u)$ is the derivative of the Weierstrass P-function, and $\varkappa$ is the first coefficient in the expansion of W at the infinite point $\infty \in \mathcal{E}$: $W = (\varkappa + O(u))du$, that is,

$$\varkappa = \lim_{z \to \infty} \frac{\bar{q}_k(z)}{\bar{Q}_k(z)} = b_k.$$

Proof. In view of $2\sqrt{R_3(z_i)} = \wp'(w_i)$, the condition $\mathrm{Res}_{\mathcal{P}_i^-} \bar{W} = -1$ is equivalent to (9.10).

It is known ([8]) that the above integral has the form

$$\int_{\infty}^{P} W = \log \frac{\sigma(u - w_1)\cdots\sigma(u - w_k)}{\sigma(u + w_1)\cdots\sigma(u + w_k)} + C_1 u + C_0, \qquad C_1, C_0 = \text{const}.$$

So, it remains to calculate C_1 for the differential (9.8). Differentiate both parts of (9.9) by u and evaluate the result at $u = 0$ ($z = \infty$). Then the right hand side gives[4]

$$\sum_{i=1}^{k} \left[\frac{\sigma'(u - w_i)\sigma(u + w_i) - \sigma'(u + w_i)\sigma(u - w_i)}{\sigma(u - w_i)\sigma(u + w_i)} + 2\zeta(w_i) \right]_{u=0} + \varkappa$$
$$= \sum_{i=1}^{k} [\zeta(u - w_i) - \zeta(u + w_i) + 2\zeta(w_i)]_{u=0} + \varkappa = \varkappa.$$

Derivation of the left hand side of (9) gives

$$\left(\frac{d}{dz} \int_{\infty}^{P=(z,w)} W \right) \frac{dz}{du}\bigg|_{u=0} = \lim_{z \to \infty} \frac{\bar{q}_k(z)}{\bar{Q}_k(z)},$$

[4]Here we used $\zeta(u) = \sigma'(u)/\sigma(u)$ and oddness of $\zeta(u)$.

which is precisely b_k. $\square$

Under a birational transformation $(z, w) \to (\lambda, \mu)$, which sends $z = \infty$ to $\lambda = c$ and converts $\mathcal{E}$ to the even order curve (3.2),

$$\mu^2 = U_4(\lambda) = -(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c),$$

the differential (9.8) takes the form

$$W = \frac{K_k(\lambda)}{Q_k(\lambda)} \frac{d\lambda}{\sqrt{U_4(\lambda)}}$$

with certain degree k polynomials $K_k(\lambda)$ and $Q_k(\lambda) = r_0(\lambda - r_1)\cdots(\lambda - r_k)$. Then Theorem 9 implies

$$\int_{(c,0)}^{P=(\lambda,\mu)} \frac{q_k(\lambda)}{Q_k(\lambda)} \frac{d\lambda}{\sqrt{U_4(\lambda)}} =$$

$$= \log \frac{\sigma(u - w_1)\cdots\sigma(u - w_k)}{\sigma(u + w_1)\cdots\sigma(u + w_k)} + 2[\zeta(w_1) + \cdots + \zeta(w_k)]u + \delta\, u, \tag{9.11}$$

$$w_k = \int_c^{(r_k, \sqrt{R_4(r_k)})} \frac{d\lambda}{\sqrt{U_4(\lambda)}}\, d\lambda, \quad \delta = \frac{K_k(c)}{Q_k(c)}, \tag{9.12}$$

which is the expression (5.1) in Proposition 5. Under the birational transformation (3.5), the condition (9.10) takes the form (5.4). $\square$

Proof of Theorem 6 .

1). The structure of the solutions (5.5) follows from Theorem 4 and Proposition 5. Namely, substituting the sigma function expressions (5.2), (5.1) for each $F_{s,\alpha}(\lambda)$ and W_α into (4.1), one obtains these solutions.

Now notice that, in view of the leading behavior (3.14), the Laurent expansions of (5.5) near $u = 0$ and $t = t_0$ are

$$G_\alpha^{(1,2)} = \frac{\mathrm{i}\,\epsilon_\alpha}{u^k} + O(u^{-k+1}) = (-1)^{(k-1)/2} \frac{\epsilon_i}{n^k (t - t_0)^k} + O((t - t_0)^{-k+1}).$$

The leading terms are proportional to those of the required expansions (3.26), hence the constant factors in the components of $G_\alpha^{(1,2)}$ are correct.

The structure of $G_\alpha^{(1,2)}(u)$ implies that they are elliptic functions of the second kind. Since all the solutions of Poisson equations are single-valued, Theorem 2 is fully applicable, hence the vectors $G^{(1,2)}(u)$ must have the vector monodromy (5.7) with certain factors s_j^2. Then

$$G_\alpha^{(1)}(u + 2\Omega_j) = \pm s_j\, G_\alpha^{(1)}(u), \quad G_\alpha^{(2)}(u + 2\Omega_j) = \pm s_j^{-1}\, G_\alpha^{(2)}(u),$$

On the other hand, from the quasiperiodicity law of $\sigma(u)$ we have, in particular,

$$G_1^{(1)}(u + 2\Omega_1) = s_1\, G_1^{(1)}(u), \quad G_1^{(2)}(u + 2\Omega_1) = s_1^{-1}\, G_1^{(2)}(u), \tag{9.13}$$
$$s_1 = \exp(2\Theta_1\Omega_1 - 2\Sigma_1\eta_1).$$

By the construction of the vectors $G_\alpha^{(1,2)}$ (see (4.6)), for any $u \in \mathbb{C}$

$$G_1^{(1)}(u)P_1(u) + G_2^{(1)}(u)P_2(u) + G_3^{(1)}(u)P_3(u) \equiv 0\,. \tag{9.14}$$

Here, from (3.16) and (3.20), we have

$$\begin{aligned}
P(u + 2\Omega_1) &= (P_1(u), -P_2(u), -P_3(u))^T, \\
P(u + 2\Omega_2) &= (-P_1(u), P_2(u), -P_3(u))^T, \\
P(u + 2\Omega_3) &= (-P_1(u), -P_2(u), P_3(u))^T.
\end{aligned}$$

This, together with the integral (9.14), and (9.13) implies

$$G^{(1)}(u + 2\Omega_1) = (s_1 G_1^{(1)}(u), -s_1 G_2^{(1)}(u), -s_1 G_3^{(1)}(u))^T.$$

Repeating the argumentation for other Ω_j, we obtain the behavior (5.6), (5.8).

3) Let the half-period Ω_j be imaginary and Ω_2 be real. Then item (2) of Theorem 2 implies that $|s_j^2| = 1$ if we identify the periods T_1, T_2 with some of full periods $4\Omega_1, 4\Omega_2, 4\Omega_3$ of $M_i(u)$. Hence, also $|s_j| = 1$. By (5.8), $s_j = \pm \exp(2\Theta_i \Omega_j - 2\Sigma_i \eta_j)$, therefore the argument $2\Theta_i\Omega_j - 2\Sigma_i\eta_j$ is imaginary. Since η_j is imaginary and η_2 is real, this means that $2\Theta_i\Omega_2 - 2\Sigma_i\eta_2$ is real, and s_2 is real.

Next, from (5.6), we get, for example,

$$\frac{G_1^{(1)}}{G_3^{(1)}}(u + 2\Omega_1) = -\frac{G_1^{(1)}}{G_3^{(1)}}(u), \quad \frac{G_1^{(1)}}{G_3^{(1)}}(u + 2\Omega_2) = \frac{G_1^{(1)}}{G_3^{(1)}}(u). \tag{9.15}$$

Hence, $G_1^{(1)}(u)/G_3^{(1)}(u)$ is an elliptic function with the periods $4\Omega_1, 2\Omega_2$. Its zeros and poles in the parallelogram of periods are

$$\{w_{1,1}, \dots, w_{k,1}, w_{1,1} + 2\Omega_1, \dots, w_{k,1} + 2\Omega_1\} \quad \text{and, respectively,} \\ \{w_{1,3}, \dots, w_{k,3}, w_{1,3} + 2\Omega_1, \dots, w_{k,3} + 2\Omega_1\}.$$

Then, according to the Abel theorem, the difference of their sums must be zero modulo the lattice $\{4\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\}$:

$$2\Sigma_1 - 2\Sigma_3 \in \{4\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\},$$

therefore, $\Sigma_1 - \Sigma_3 \in \{2\Omega_1\mathbb{Z} + \Omega_2\mathbb{Z}\}$. Further, the case $\Sigma_1 - \Sigma_3 \in \{2\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\}$ is not possible (otherwise $G_1^{(1)}(u)/G_3^{(1)}(u)$ would had been a product of an elliptic function with the period lattice $\{2\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\}$ and an exponent, i.e., not doubly-periodic itself). Hence,

$$\Sigma_1 - \Sigma_3 \equiv \Omega_2 \mod \{2\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\}.$$

Applying the same argumentation to the quotients $G_2^{(1)}(u)/G_3^{(1)}(u)$, $G_1^{(1)}(u)/G_2^{(1)}(u)$, we arrive at the first part of relations (5.9).

Next, the quasi-periodicity of $\sigma(u)$ implies

$$\frac{G_1^{(1)}}{G_3^{(1)}}(u+2\Omega_1) = \exp\left[2(\Sigma_1-\Sigma_3)\Omega_1 - 2(\Theta_1-\Theta_3)\eta_1\right]\frac{G_1^{(1)}}{G_3^{(1)}}(u),$$
$$\frac{G_1^{(1)}}{G_3^{(1)}}(u+2\Omega_2) = \exp\left[2(\Sigma_1-\Sigma_3)\Omega_2 - 2(\Theta_1-\Theta_3)\eta_2\right]\frac{G_1^{(1)}}{G_3^{(1)}}(u).$$

Comparing this with (9.15) and using $\Sigma_1-\Sigma_3 \equiv \Omega_2$, as well as the known Legendre relations

$$\eta_2\Omega_3-\eta_3\Omega_2 = \eta_3\Omega_1-\eta_1\Omega_3 = \eta_1\Omega_2-\eta_2\Omega_1 = \frac{1}{2}\pi\,\mathrm{i}\,,$$

we get the second half of (5.9).

4). Since $G^{(1)}(u)$ is a solution of the Poisson equations, it must satisfy the integral

$${G_1^{(1)}}^2(u)+{G_2^{(1)}}^2(u)+{G_3^{(1)}}^2(u) = M.$$

Then

$$\begin{aligned}&{G_1^{(1)}}^2(u+2n\Omega_2)+{G_2^{(1)}}^2(u+2n\Omega_2)+{G_3^{(1)}}^2(u+2n\Omega_2) = \\ &\quad = s_2^{2n}\left[{G_1^{(1)}}^2(u)+{G_2^{(1)}}^2(u)+{G_3^{(1)}}^2(u)\right] = M \qquad \forall\, n\in\mathbb{Z}.\end{aligned}$$

Since $|s_2|\neq 1$, letting above $n\to\infty$ or $n\to-\infty$, we conclude that M must be zero. The same argumentation applied to $G^{(2)}(u)$, give the second identity in (5.10).

Let now Ω be a half-period of E, $\eta=\zeta(\Omega)$ and $w\in\mathbb{C}$ an arbitrary number. From the identity

$$\frac{\sigma(\Omega+w)\,\sigma(\Omega-w)}{\sigma^2(\Omega)\,\sigma^2(w)} = \wp(\Omega)-\wp(w),$$

using the quasiperiodicity of $\sigma(u)$, we get

$$-\,e^{2\eta\,w}\frac{\sigma^2(\Omega-w)}{\sigma^2(\Omega)\,\sigma^2(w)} = \wp(\Omega)-\wp(w). \tag{9.16}$$

Applying this to the solutions (5.5) in the case $\Omega=\Omega_i, w=w_{l,i}$ and using (5.8) gives us

$$\begin{aligned}\left[G_\alpha^{(1)}(\Omega_j)\right]^2 &= \epsilon_\alpha^2\prod_{l=1}^k e^{2\zeta(w_{\alpha,l})\Omega_j}\frac{\sigma^2(\Omega_j-w_{\alpha,l})}{\sigma^2(\Omega_j)\,\sigma^2(w_{\alpha,l})} = (-1)^{\delta_{\alpha j}}\epsilon_\alpha^2\, s_j e^{2\Sigma_\alpha\eta_j}\prod_{l=1}^k\frac{\sigma^2(\Omega_j-w_{\alpha,l})}{\sigma^2(\Omega_j)\,\sigma^2(w_{\alpha,l})}\\ &= (-1)^{\delta_{\alpha j}}\epsilon_\alpha^2 s_j\prod_{l=1}^k e^{2w_{\alpha,l}\,\eta_j}\frac{\sigma^2(\Omega_j-w_{\alpha,l})}{\sigma^2(\Omega_j)\,\sigma^2(w_{\alpha,l})}\\ &= -(-1)^{\delta_{\alpha j}}\epsilon_i^2\, s_j\cdot(\wp(\Omega_j)-\wp(w_{\alpha,1}))\cdots(\wp(\Omega_j)-\wp(w_{\alpha,k})),\end{aligned}$$

i.e., the expressions (5.12) for $G_i^{(1)}(\Omega_j)$. The expressions for $G_i^{(2)}(\Omega_j)$ are obtained in the same way.

Next, the structure of the solutions (5.5) gives

$$\sum_{\alpha=1}^{3} G_{\alpha}^{(1)}(u) G_{\alpha}^{(2)}(u) = -\sum_{\alpha=1}^{3} \epsilon_{\alpha}^{2} \prod_{l=1}^{k} (\wp(u) - \wp(w_{l,\alpha})). \tag{9.17}$$

On the other hand, as follows from (5.10) and (5.12), for any $\alpha = 1, 2, 3$

$$\epsilon_{\alpha}^{2} \prod_{l=1}^{k} (\wp(\Omega_{\alpha}) - \wp(w_{l,\alpha})) = \epsilon_{\beta}^{2} \prod_{l=1}^{k} (\wp(\Omega_{\alpha}) - \wp(w_{l,\beta})) + \epsilon_{\gamma}^{2} \prod_{l=1}^{k} (\wp(\Omega_{\alpha}) - \wp(w_{l,\gamma})),$$
$$(\alpha, \beta, \gamma) = (1, 2, 3).$$

Then, for $u = \Omega_j$, $j = 1, 2, 3$, relation (9.17) yields

$$-\sum_{\alpha=1}^{3} G_{\alpha}^{(1)}(\Omega_j)\, G_{\alpha}^{(2)}(\Omega_j) = 2\epsilon_j^2 \prod_{l=1}^{k} (\wp(\Omega_j) - \wp(w_{l,j}))\,.$$

Since $\langle G^{(1)}(u), G^{(2)}(u) \rangle$ is a first integral, we get the relations (5.13).

It remains to prove that the latter equal Π. According to Theorem 4 and relation (4.3),

$$Q_{k,\alpha}(\lambda) = r_{0,\alpha} \prod_{l=1}^{k} (\lambda - r_{i,\alpha}) = \text{const}(P_{\beta}^{2}(\lambda) + P_{\gamma}^{2}(\lambda)) \cdot (\lambda - c)^k$$

for $(\alpha, \beta, \gamma) = (1, 2, 3)$. This implies, in particular,

$$C\,(\wp(u) - \wp(w_{1,1})) \cdots (\wp(u) - \wp(w_{1,k})) = P_2^2(u) + P_3^2(u) \tag{9.18}$$

with a certain constant C. On the other hand, in view of the solutions (3.22), one obtains,

$$P_2^2(u) + P_3^2(u) = \epsilon_2^2\,(\wp(u) - \wp(\Omega_2)) \prod_{l=1}^{s} (\wp(u) - \wp(v_{l,2}))^2$$
$$+\, \epsilon_3^2\,(\wp(u) - \wp(\Omega_3)) \prod_{l=1}^{s} (\wp(u) - \wp(v_{l,3}))^2\,.$$

Letting in the above two relations $u \to \infty$, taking into account the expansion $\wp(u) = 1/u^2 + O(1)$, and comparing the leading coefficients of $1/u^{2k}$, we find

$$C = \epsilon_2^2 + \epsilon_3^2 = \frac{1}{(a_2 - a_1)(a_2 - a_3)} + \frac{1}{(a_3 - a_1)(a_3 - a_1)} = -\epsilon_1^2.$$

Since $P_1(u = \Omega_1) = 0$, we have $\Pi = P_1^2(u) + P_2^2(u) + P_3^2(u) = P_2^2(\Omega_1) + P_3^2(\Omega_1)$. Comparing this with (9.18) for $u = \Omega_1$, we get

$$\Pi = -\epsilon_1^2 \prod_{l=1}^{k} (\wp(\Omega_1) - \wp(w_{l,1}))\,,$$

and, therefore, the last equality in (5.13). $\square$

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